A Convincing Argument (but not a proof) as to why the Fundamental Theorem of Caluclus should be true

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1 The Fundamental Theorem of calculus

The FTC has two parts, the most important one saying that if f is a differentiable function on $[a, b]$, then it is integrable on $[a, b]$ and:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

You might have heard it in the form $\int_a^b f(x)dx = F(b) - F(a)$, which is essentially the same, but we will use the above form to give a *convincing argument* why the FTC should be true. Again, it's just a convincing argument, as by no means a proof! It is meant to convince you that the FTC makes sense!

2 Convincing Argument

We will not bother about issues of integrability of f', because it's very technical and has lots of epsilons involved. Instead, we will show intuitively why the above formula holds.

For this means, we will use the left-hand-sums L_n and will let $n \to \infty$ (**once** we know that a function is integrable, we can calculate its integral in any way we'd like!). Now by definition of the left-hand-sums applied to f', we get:

$$L_n = \frac{b-a}{n} \sum_{i=0}^{n-1} f'(x_i)$$

Note: If you don't like a and b, assume $b = 1$ and $a = 0$. The argument is exactly the same, but there are fewer constants!

But now, by definition, for each i,

$$f'(x_i) = \lim_{h \to 0} \frac{f(x_i + h) - f(x_i)}{h}$$

Now, since we know that the limit exists, we can say that for h small, the above equality approximately holds, i.e.

$$f'(x_i) \approx \frac{f(x_i + h) - f(x_i)}{h}$$

In particular, it should hold for $h = \frac{b-a}{n}$ when n is large (again, this is not a proof, and one would need to be rigorous about it). So $\frac{1}{h} = \frac{n}{b-a}$, so we get:

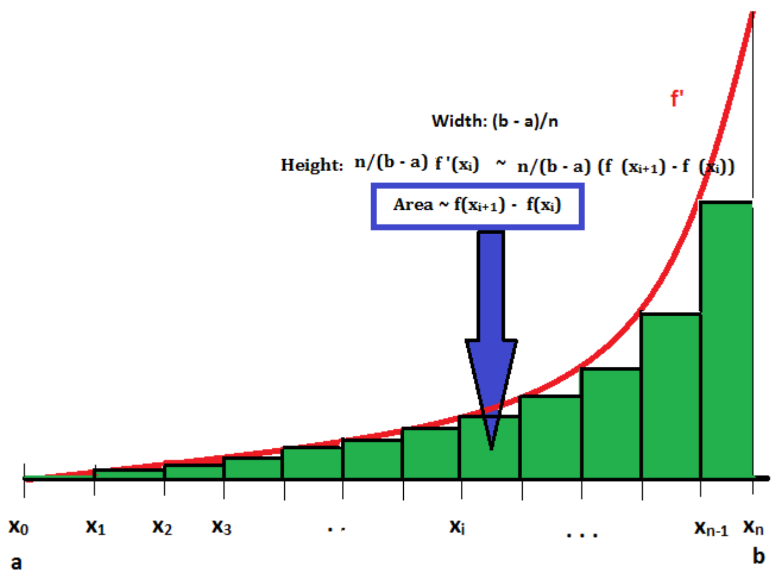
$$f'(x_i) \approx \frac{n}{b-a} \left(f(x_i + \frac{b-a}{n}) - f(x_i) \right)$$

However, by definition, $x_i + \frac{b-a}{n} = x_{i+1}$, so we get:

$$f'(x_i) \approx \frac{n}{b-a} \left(f(x_{i+1}) - f(x_i) \right)$$

A picture might help you understanding the situation:

1A/FTC.png



Finally, we get:

$$
\begin{aligned}
L_n =& \frac{b-a}{n}\sum_{i=0}^{n-1} f'(x_i) \\
\approx& \frac{b-a}{n}\sum_{i=0}^{n-1} \frac{n}{b-a}\left(f(x_{i+1}) - f(x_i)\right) \\
=& \left(\frac{b-a}{n}\right)\left(\frac{n}{b-a}\right)\sum_{i=0}^{n-1}\left(f(x_{i+1}) - f(x_i)\right) \\
=& \sum_{i=0}^{n-1}\left(f(x_{i+1}) - f(x_i)\right) \\
=& f(x_1) - f(x_0) + f(x_2) - f(x_1) + f(x_3) - f(x_2) + \cdots + f(x_{n-1}) - f(x_{n-2}) + f(x_n) - f(x_{n-1})
\end{aligned}
$$

But now notice that all the terms cancel out **except** $f(x_0) = f(a)$ and $f(x_n) = f(b)$! (this is called a **telescoping sum**, and you'll see more of those in Math 1B).

So we get:

$$L_n = f(x_n) - f(x_0) = f(b) - f(a)$$

But $f(b) - f(a)$ does not depend on n, and so $\lim_{n\to\infty} L_n = f(b) - f(a)$

$$\int_a^b f'(x)dx = \lim_{n\to\infty} L_n = f(b) - f(a)$$

And we're done! :)